ON CR (CAUCHY-RIEMANN) ALMOST COSYMPLECTIC MANIFOLDS

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Abstract. In the paper we develop a framework for the alternative way of the study of a local geometry of almost cosymplectic manifolds with Kälerian leaves. The main idea is to apply the concept of a geometry and analysis of CR manifolds. Locally the almost cosymplectic manifold is modeled on the 'mixed' space $\mathbb{R} \times \mathbb{C}^n$. There is given a complete local description of the underlying almost contact metric structure in the system of local, mixed - real, complex-coordinates. We also introduce a notion of a canonical Hermitian complex connection in the CR structure of a CR almost cosymplectic manifold. As an example we provide detailed descritpion of almost cosymplectic $(-1, \mu, 0)$-spaces.

Dedicated to the Memory of Marek Kucharski

1. Introduction

The paper is thought of as a preliminary to the subject of possible applications of the methods coming from the geometry of CR manifolds or complex geometry. The starting point is a notion of a CR manifold.

It is very common nowadays in the geometry of the almost contact metric manifold to impose conditions of a "CR integrability" of an almost contact metric structure. One of the first result concerning "CR integrable" almost contact metric structures is the theorem of S. Tanno - the characterization of a CR integrable contact metric manifold [19].

In the settings of almost cosymplectic manifolds CR geometry implicitly appeared in the time when Z. Olszak introduced quite naturally the notion of the almost cosymplectic manifolds with Kählerian leaves [16]. Now it is clear that these manifolds are exactly CR integrable almost cosymplectic manifolds. And this statement is almost tautological from the point of view of the geometry of CR manifolds. However the main focus was on studying the Riemannian geometry using tensor calculus.

In the presented paper we propose an alternative way to study manifolds with Kählerian leaves, based on the CR geometry. We hope that this alternative allow us to solve some long-standing and yet unsolved problems. For example we do not know are there exist almost cosymplectic non-cosymplectic manifolds of a pointwise-constant φ-sectional curvature in dimensions > 3? Even in the case of manifold with Kählerian leaves the answer is unknown. The other benefit is that new classes of manifolds appear quite naturally on the base of the local description in the so-called CR charts (Sec. 3). The main point is that locally an almost cosymplectic manifold is modeled on $\mathbb{R} \times \mathbb{C}^n$ thus we have a 'mixed' local coordinates and an almost contact metric structure (φ, ξ, η, g) can be described using these 'mixed' coordinates. As a working example we provide such description in very details for almost cosymplectic $(-1, \mu, 0)$-spaces with $\mu = const$.

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2. Preliminaries

2.1. **Almost cosymplectic manifolds.** A manifold $\mathcal{M}$ of an odd dimension, $\dim \mathcal{M} = 2n+1 \geqslant 3$ smooth, connected, being endowed with an almost contact metric structure (φ, ξ, η, g), where φ is a $(1,1)$-tensor field, ξ a vector field, η a 1-form and g a Riemannian metric, and the following conditions are satisfied [4]

$$\begin{aligned} &\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi), \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{1}$$

is called an almost contact metric manifold. If the forms η and the fundamental skew-symmetric 2-form $\Phi(X, Y) = g(\varphi X, Y)$ are both closed the almost contact metric manifold $\mathcal{M}$ is called almost cosymplectic [11, 15].

An almost contact metric manifold is called normal if [4]

$$N_\varphi + 2d\eta \otimes \xi = 0 \tag{2}$$

here N_φ denotes the Nijenhuis torsion of φ defined by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

The normal almost cosymplectic manifold is called cosymplectic. From (2) it follows that an almost cosymplectic manifold is cosymplectic if and only if the torsion N_φ vanishes. In that case the tensor field φ is integrable as a G-structure, i.e. there is a suitable atlas of local coordinates charts: the local coefficients of φ on each chart are constant functions. Arbitrary cosymplectic manifold is locally a Riemannian product of a real line (an open interval) and a Kähler manifold. D. E. Blair proved [3] that an almost contact metric manifold is cosymplectic if and only if

$$\nabla \varphi = 0,$$

for the Levi-Civita connection ∇. S. I. Goldberg and K. Yano studying harmonic forms proved the following theorem [11]: an almost cosymplectic manifold is cosymplectic if and only if

$$R(X, Y)\varphi Z = \varphi R(X, Y)Z, \tag{3}$$

R the Riemann curvature operator

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

That theorem can be viewed as an analogue of a similar theorem for Kähler manifolds. Precisely for an almost cosymplectic manifold (3) implies $\nabla\varphi = 0$ and by the Blair result the manifold is cosymplectic.

An almost cosymplectic manifold $\mathcal{M}$ carries a canonical foliation $\mathcal{M} = \bigcup_{p \in \mathcal{M}} \mathcal{N}_p$ which corresponds to a completely integrable distribution defined by $\eta = 0$. A leaf $\mathcal{N}_p \subset \mathcal{M}$ can be in a natural way considered as an almost Kähler manifold $(\mathcal{N}_p, J, G)$. The almost Hermitian structure (J, G) is given by

$$\iota_* \circ J = \varphi \circ \iota_*, \qquad G = \iota^* g,$$

ι denotes the inclusion map $\iota : \mathcal{N}_p \subset \mathcal{M}$. If $\mathcal{M}$ is cosymplectic then all leaves are in fact Kähler manifolds. However the converse is not true. Z. Olszak introduced almost cosymplectic manifolds defined by imposing the following geometric condition: each leaf is a Kähler manifold. We have the following characterization [16]: an almost cosymplectic manifold has Kählerian leaves if and only if

$$(\nabla_X \varphi)Y = -g(\varphi AX, Y)\xi + \eta(Y)\varphi AX, \tag{4}$$

a $(1,1)$-tensor field A is defined by

$$AX = -\nabla_X \xi. \tag{5}$$

If the curvature operator of the Levi-Civita connection $R(X,Y)Z$ of an almost cosymplectic manifold satisfies

$$R(X,Y)\xi = \eta(Y)PX - \eta(X)PY, \qquad P = \kappa Id + \mu h + \nu A \tag{6}$$

Id the identity tensor, $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, $\mathcal{L}_\xi$ the Lie derivative, and κ, μ, ν are functions such that for the 1-forms $d\kappa$, $d\mu$, $d\nu$ we have

$$d\kappa \wedge \eta = d\mu \wedge \eta = d\nu \wedge \eta = 0, \tag{7}$$

then the manifold is called an almost cosymplectic (κ,μ,ν)-space [6]. The particular case is when κ, μ, ν are constants. Almost cosymplectic manifolds satisfying the condition (6) with $\kappa = const$, $\mu = \nu = 0$ were studied in [5]; and with κ, $\mu = const.$, $\nu = 0$ in [8, 9, 10]. Manifolds with $\kappa = -1$, $\mu = const.$, $\nu = 0$ are classified in [7].

Given positive functions α, β on $\mathcal{M}$ an almost contact metric structure $(\varphi', \xi', \eta', g')$ defined by

$$\varphi' = \varphi, \quad \xi' = \beta^{-1}\xi, \quad \eta' = \beta\eta, \quad g' = \alpha g + (\beta^2 - \alpha)\eta \otimes \eta,$$

is called a D-conformal deformation of the structure (φ, ξ, η, g). The structure $(\varphi', \xi', \eta', g')$ is by itself an almost cosymplectic if and only if α is a constant and the function β satisfies $d\beta \wedge \eta = 0$.

Proposition 1. ([6]) *If $(\mathcal{M}, \varphi, \xi, \eta, g)$ is an almost cosymplectic (κ,μ,ν)-space then its image by a D-conformal deformation (assuming an image is almost cosymplectic) is a (κ', μ', ν')-space where*

$$\kappa' = \frac{\kappa}{\beta^2}, \quad \mu' = \frac{\mu}{\beta}, \quad \nu' = \frac{\nu\beta - d\beta(\xi)}{\beta^2}.$$

So, roughly speaking we may say that the class of almost cosymplectic (κ,μ,ν)-spaces is closed with respect to the group of the inner D-conformal deformations. Deformations are inner in the sense that they preserve the class of almost cosymplectic manifolds.

For an almost cosymplectic (κ,μ,ν)-space the tensor field A and the functions κ, μ and ν satisfy the relations

$$\begin{aligned} A^2 &= -\kappa(Id - \eta \otimes \xi), \\ \nabla_\xi A &= \mu\, h + \nu\, A, \\ d\kappa(\xi) &= 2\nu\kappa, \end{aligned} \tag{8}$$

and these relations are fundamental in the process of the classification of (κ,μ,ν)-spaces. Eg. they imply that if $\kappa = const \neq 0$ then the function ν must vanish identically. Another consequence is that arbitrary (κ,μ,ν)-space is (generally speaking locally) D-conformal to a $(-1,\mu,0)$-space. Thus the almost cosymplectic $(-1,\mu,0)$-spaces are of particular importance and they are called the "model spaces".

2.2. **CR manifolds (**[2, 12, 18]**).** For complex vector fields Z_1, Z_2 a bracket $[Z_1, Z_2]$ is defined as a (unique) complex vector field such that

$$[Z_1, Z_2]f = Z_1(Z_2 f) - Z_2(Z_1 f),$$

for arbitrary complex-valued smooth function f. A CR manifold is a pair $(\mathcal{M}, \mathcal{H})$ where $\mathcal{M}$ is a smooth manifold and $\mathcal{H}$ is a C^∞ complex subbundle of $TM \otimes \mathbb{C}$ such that:

$$\mathcal{H}_p \cap \bar{\mathcal{H}}_p = 0, \qquad p \in M, \tag{9}$$

and the set of sections $\Gamma(\mathcal{H})$ of $\mathcal{H}$ is closed with respect to the bracket operation.

A $m = \dim_{\mathbb{C}} \mathcal{H}$ is called a CR dimension of a CR manifold $(\mathcal{M}, \mathcal{H})$ and $n - 2m$ a CR codimension.

The examples of CR manifolds are complex manifolds (CR codimension is 0) and real hypersurfaces in $\mathbb{C}^n$ (CR codimension is 1).

Now let $(\mathcal{M},\mathcal{H})$ be a CR manifold of a hypersurface type, i.e. CR-codim = 1. For an imaginary non vanishing 1-form τ annihilating $\mathcal{H}\oplus\bar{\mathcal{H}}$ (it is possible that τ is defined only locally), and a vector $z\in\mathcal{H}_p$ let

$$z\mapsto L_p(z)=\tau_p([Z,\bar{Z}]_p), \tag{10}$$

where $Z\in\Gamma(\mathcal{H})$ is an local extension of z. The function $z\mapsto L_p(z)$ is a real quadratic form on $\mathcal{H}_p$ called the Levi form. The form L_p is defined up to a nonzero scalar for if τ is replaced by $\tau'=f\tau$, $f(p)\neq 0$ then (10) yields to $L'_p(z)=f(p)L_p(z)$. If L_p is non-degenerate it is a pseudo-Hermitian form on $\mathcal{H}_p$.

A CR-manifold $(\mathcal{M},\mathcal{H})$ is said to be Levi flat if its Levi form vanishes everywhere.

2.3. **CR structure of an almost cosymplectic manifold.** Let $\mathcal{M}$ be an almost contact metric manifold and $\mathcal{D}$ a distribution $\mathcal{D}=Im\,\varphi$, therefore $\mathcal{D}$ is a field of hyperplanes on $\mathcal{M}$

$$\mathcal{D}:\mathcal{M}\ni p\mapsto\mathcal{D}_p=\{x\in T_p\mathcal{M}: x=\varphi y \text{ for } y\in T_p\mathcal{M}\}.$$

Equivalently $\mathcal{D}$ can be defined as the kernel of the form η. We always have $\mathrm{Ker}\,\eta=Im\,\varphi$ The complexification $\mathcal{D}\otimes\mathbb{C}$ splits into a direct sum $\mathcal{D}'\oplus\mathcal{D}''$ of $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of the (complexified) tensor field φ. For a section $Z\in\Gamma(\mathcal{D}')$ let $X=\dfrac{1}{2}(Z+\bar{Z})$. The conditions $\varphi Z=\sqrt{-1}Z$, $\eta(\varphi Z)=0$ together imply that

$$Z=X-\sqrt{-1}\,\varphi X,\quad \eta(X)=0. \tag{11}$$

Conversely given a real vector field X, $\eta(X)=0$ (11) defines a section of $\mathcal{D}'$.

Let suppose that the pair $(\mathcal{M},\mathcal{D}')$ is an almost cosymplectic CR manifold, i.e. $\mathcal{M}$ is an almost cosymplectic manifold and the complex distribution $\mathcal{D}'$ defined as above is formally involutive. In order to compute the Levi form L we can take $\tau=-\sqrt{-1}\,\eta$ in (10)

$$L_p(z)=-\sqrt{-1}\,\eta([Z,\bar{Z}]_p)=2\,\eta([X,\varphi X]_p),$$

where $Z=X-\sqrt{-1}\,\varphi X$ and $Z_p=z$. Since $d\eta=0$ and $\eta(X)=\eta(\varphi X)=0$ it follows that

$$L_p(z)=-4\,d\eta(X_p,\varphi X_p)=0.$$

Therefore L vanishes identically and $(\mathcal{M},\mathcal{D}')$ is a Levi flat CR manifold (of the hypersurface type).

Intuitively: a Levi flat CR-manifold is foliated by a family of (real) hypersurfaces, each hypersurface admits a complex structure and this complex structure varies when passing from a hypersurface to another hypersurface according to a differentiability class of the manifold. From this of point view, having in mind the definition of almost cosymplectic manifold with Kählerian leaves, the next proposition is almost tautological

Proposition 2. *The pair $(M,\mathcal{D}')$ is a (Levi flat) CR manifold if and only if M has Kählerian leaves.*

In other words an almost cosymplectic CR manifold is exactly the same concept as an almost cosymplectic manifold with Kählerian leaves.

3. Analytic almost cosymplectic CR manifolds. The canonical Hermitian structure.

3.1. **CR charts on an analytic almost cosymplectic CR manifold.** Suppose $\mathcal{M}$ is a real-analytic almost cosymplectic CR manifold (or a manifold with Kälerian leaves), and the tensor fields φ, ξ, η and g are assumed to be real-analytic. In a consequence the CR structure $\mathcal{D}'$ of

$\mathcal{M}$ is real-analytic - there are local real-analytic sections spanning $\mathcal{D}'$. Now, according to the theorem of A. Andreotti and D.C. Hill ([1]) there is a local embedding (real-analytic)

$$f : \mathcal{M} \to \mathbb{C}^{n+1}, \tag{12}$$

($\dim \mathcal{M} = 2n+1$) such that $f(\mathcal{M})$ is locally a real-analytic hypersurface. If

$$\mathcal{M} \ni q \mapsto p = (z^1, \dots, z^n, z^{n+1}) = f(q) \in f(\mathcal{M}),$$

then there is a maximal complex subspace $\mathcal{H}_p$ in $T_p^{(1,0)}\mathbb{C}^{n+1}$ such that

$$\Re(\mathcal{H}_p \oplus \overline{\mathcal{H}}_p) = J(T_p f(\mathcal{M})) \cap T_p f(\mathcal{M}),$$

J denotes the canonical complex structure of $\mathbb{C}^{n+1}$, and the complexification of the tangent map f_* defines a $\mathbb{C}$-linear isomorphism between complex spaces

$$: \quad \mathcal{D}'_q \xrightarrow{f_*^{\mathbb{C}}} \mathcal{H}_p.$$

If $\mathcal{U}_q \subset \mathcal{M}$ is sufficiently small then $f|_{\mathcal{U}_q}$ is a diffeomorphism onto its image $f(\mathcal{U}_q)$. Now let consider a real hypersurface in $\mathbb{C}^{n+1}$. That is a set of zeros $\mathcal{S} = r^{-1}(0)$ of a smooth real-valued function $r : \mathbb{C}^{n+1} \to \mathbb{R}$. We assume that r is regular: $dr \neq 0$ for each point of $\mathcal{S}$. The space $\mathbb{C}^{n+1}$ endowed with its canonical flat Kähler metric becomes an Euclidean space $\mathbb{E}^{2n+2}$ so let H be the second fundamental form of $\mathcal{S}$ treated as a hypersurface in $\mathbb{E}^{2n+2}$. Let define a real quadratic differential form on a complex subbundle $\mathcal{H} : p \mapsto \mathcal{H}_p$, $p \in \mathcal{S}$ ($\mathcal{H}_p$ is defined similarly as above for $f(\mathcal{M}) = \mathcal{S}$) by the formula

$$L(z) = H(X,X) + H(JX,JX), \qquad z \in \mathcal{H}_p,\ z = X - \sqrt{-1}JX,\ X \in T_p\mathcal{S},$$

for a complex vector z tangent to $\mathcal{S}$. The form L is a Levi form of the real hypersurface $\mathcal{S}$ (cf. [18]). Let consider two examples

1. $\mathcal{S} = \mathbb{S}^{2n+1}(r)$ a canonical sphere of the radius r; as the second fundamental form is non-degenerate and definite the Levi form is nondegenerate and definite: $\mathbb{S}^{2n+1}(r)$ is an example of a strictly pseudo-convex real hypersurface,
2. $\mathcal{S} = \{p = (z^1, \dots, z^{n+1}) \in \mathbb{C}^{2n+1} : \Im z^{n+1} = 0\}$; clearly $\mathcal{S}$ now is simply a hyperplane hence $H = 0$ identically, in a consequence the Levi form vanishes; $\mathcal{S}$ is the simplest example of a Levi flat real hypersurface.

We need the following basic result: a Levi flat real-analytic hypersurface is locally biholomorphic to the hyperplane described in the example (2). So let assume that a Levi flat real hypersurface is passing through the origin o of $\mathbb{C}^{n+1}$ then there is a small disk D_o centered at o and a biholomorphism $F : D_o \to D_o$ such that

$$F(\mathcal{S} \cap D_o) = \{\Im z^{n+1} = 0\} \cap D_o.$$

Now let consider the sequence of maps

$$\mathcal{U}_q \xrightarrow{f} f(\mathcal{U}_q) \cap D_o \xrightarrow{F} \{\Im z^{n+1} = 0\} \cap D_o, \tag{13}$$

the existence of F follows from the fact that $f(\mathcal{U}_q)$ is a real hypersurface (for sufficiently small $\mathcal{U}_q$) as it is defined above, i.e. there is a regular real function r such that $f(\mathcal{U}_q) \subset r^{-1}(0)$ and $f(\mathcal{U}_q)$ is Levi flat.

Proposition 3. *Let $(\mathcal{M}, \mathcal{D}')$ be a real-analytic almost cosymplectic CR manifold. Then each point q of $\mathcal{M}$ admits a neighborhood $\mathcal{U}_q$ and a local diffeomorphism $f_q : \mathcal{U}_q \to (-a,a) \times D'$ where $(-a,a)$ is an open interval $a > 0$ and D' is a small disk in $\mathbb{C}^n$.*

Proof. By the Andreotti, Hill theorem there is a local embedding f such that $f(\mathcal{U}_q)$ is a Levi flat real-analytic hypersurface in $\mathbb{C}^{n+1}$. Now (we may assume $f(q) = o$) there is a local biholomorphism F of an open disk D_o which maps $f(\mathcal{U}_q) \cap D_o$ onto $\{\Im z^{n+1} = 0\} \cap D$. So an image of $\mathcal{U}_q$ by the composition $F \circ f$ is contained in the hyperplane $\Im z^{n+1} = 0$. In the natural manner we identify $\Im z^{n+1} = 0$ with $\mathbb{R} \times \mathbb{C}^n$

$$\{\Im z^{n+1} = 0\} \ni p = (z^1, \dots, z^{n+1}) \mapsto (t, z^1, z^2, \dots, z^n), \quad t = \Re z^{n+1}.$$

The differential $(F \circ f)_*$ is non-degenerate at q. By the standard inverse function arguments we can assert that there is an open set of the form $(-a, a) \times D' \subset (F \circ f)(\mathcal{U}_q)$ with well-defined an inverse map $(-a, a) \times D' \to \mathcal{U}_q$. An image $\mathcal{U}'_q$ by this inverse map is the required neighborhood and we set $f_q = F \circ f|_{\mathcal{U}'_q}$. □

The CR structure of $(-a, a) \times D'$ is defined by $T^{(1,0)}D'$ - the complex bundle of $(1,0)$ vector fields on D' in natural way embedded into the complex tangent bundle of $(-a, a) \times D'$. The family $(\mathcal{U}_q, f_q)$, $q \in \mathcal{M}$ defines a very particular atlas on $\mathcal{M}$. So the manifold is covered by the coordinates charts of the form

$$(t, z^1, z^2, \dots, z^n) : (-a, a) \times D' \to \mathcal{U}_q, \quad t \in \mathbb{R}, z^i \in \mathbb{C}.$$

The transition functions

$$f_{qp} = f_q \circ f_p^{-1} : f_p(\mathcal{U}_p \cap \mathcal{U}_q) \to f_q(\mathcal{U}_p \cap \mathcal{U}_q)$$

(we verify this directly) are given as follows

$$\begin{aligned} f_{qp} \ &: \ (t, z^1, \dots, z^n) \mapsto (t', z'^1, \dots, z'^n), \\ t' &= t'(t), \\ z'^1 &= z'^1(t, z^1, \dots, z^n), \\ &\dots \\ z'^n &= z'^n(t, z^1, \dots, z^n), \end{aligned}$$

If $Z_1, \dots, Z_n$ are sections of $\mathcal{D}'$ and such that in the local coordinates on $\mathcal{U}_q$, Z_i's are 'base' vector fields:

$$Z_1 = \frac{\partial}{\partial z'^1}, \ \dots \ , Z_n = \frac{\partial}{\partial z'^n},$$

in the local coordinates on $\mathcal{U}_p$, Z_i's have the decompositions:

$$\begin{aligned} Z_1 &= \textstyle\sum_{i=1}^n f_1^i \frac{\partial}{\partial z^i}, \\ &\dots \\ Z_n &= \textstyle\sum_{i=1}^n f_n^i \frac{\partial}{\partial z^i} \end{aligned}$$

f_j^i are complex functions, therefore on $\mathcal{U}_q \cap \mathcal{U}_p$

$$\begin{aligned} \frac{\partial}{\partial z'^1} &= \sum_{i=1}^n f_1^i \frac{\partial}{\partial z^i}, \\ &\dots \\ \frac{\partial}{\partial z'^n} &= \sum_{i=1}^n f_n^i \frac{\partial}{\partial z^i}. \end{aligned}$$

These identities imply

$$\frac{\partial \bar{z}^j}{\partial z'^i} = 0, \quad \frac{\partial t}{\partial z'^j} = 0, \quad i, j = 1, \dots, n.$$

hence $\dfrac{\partial z^j}{\partial \bar{z}'^i} = 0$ and as t is a real-valued function $\dfrac{\partial t}{\partial \bar{z}'^j} = 0$.

3.2. **A local description of an almost contact metric structure.** Once we have a clear idea how to understand a 'complex coordinate' on an almost cosymplectic CR manifold we may describe the almost cosymplectic structure (φ, ξ, η, g) in terms of these complex coordinates in the way similar as it is done in the complex geometry of Hermitian manifolds. Here we follow (with necessary changes) the monograph [14].

From now on we will consider *complexified* structure $(\varphi^{\mathbb{C}}, \xi^{\mathbb{C}}, \eta^{\mathbb{C}}, g^{\mathbb{C}})$ however we use the same notation for both the structure and its complexification. It should be clear from the context when the structure is in fact the complexification. For example $\xi^{\mathbb{C}}$ is a complex vector field. i.e. a section of $T_{\mathbb{C}}\mathcal{M} = T\mathcal{M} \otimes \mathbb{C}$. The following convention is assumed: indices matching lowercase Latin letters $i, j, k, l, \ldots$ run from 1 to n, while Latin capitals $A, B, C, \ldots$ run through $0, 1, \ldots, n, \bar{0}, \bar{1}, \ldots, \bar{n}$, moreover $\bar{0} = 0$. We set

$$Z_0 = \frac{\partial}{\partial t}, \qquad Z_1 = \frac{\partial}{\partial z^1}, \ \ldots\ , Z_n = \frac{\partial}{\partial z^n}, \tag{14}$$

and

$$Z_{\bar{i}} = \overline{Z_i} = \frac{\partial}{\partial \bar{z}^i}.$$

$$\varphi Z_A = \varphi_A^B Z_B, \qquad \xi = \sum_A \xi^A Z_A, \qquad \eta_A = \eta(Z_A), \qquad g_{AB} = g(Z_A, Z_B). \tag{15}$$

Note that $g_{\bar{A}\bar{B}} = \bar{g}_{AB}$. Since the vector fields of $Z_1, \ldots, Z_n$ span $\mathcal{D}'$ we have

$$\varphi Z_i = \sqrt{-1} Z_i, \quad \varphi Z_{\bar{i}} = -\sqrt{-1} Z_{\bar{i}}. \tag{16}$$

Similarly to a Hermitian metric on a complex manifold

$$g_{i\,j} = g_{\bar{i}\,\bar{j}} = 0, \tag{17}$$

and $(g_{i\,\bar{j}})$ is a $n \times n$ Hermitian matrix $\bar{g}_{i\,\bar{j}} = g_{j\,\bar{i}}$. The coefficients $b_A = g(Z_0, Z_A)$ satisfy

$$b_{00} = \bar{b}_{00}, \quad b_{\bar{i}} = g_{0\bar{i}} = g_{\bar{0}\bar{i}} = \bar{b}_i. \tag{18}$$

Summing up all above formulas we can write

$$ds^2 = r\,dt^2 + 2\sum_{i=1}^{n} \left(b_i\,dt\,dz^i + b_{\bar{i}}\,dt\,d\bar{z}^i\right) + 2\sum_{i,j=1}^{n} g_{i\,\bar{j}}\,dz^i d\bar{z}^j, \tag{19}$$

here r stands for g_{00}. All coefficients η_A vanish except $\eta_0 = \eta(Z_0)$ therefore $\eta = udt$. Without loss of the generality we may assume $u = 1$ for $d\eta = du \wedge dt = 0$. Since $\bar{\xi} = \xi$ (ξ is real) and $\eta(\xi) = 1$ it follows that

$$\xi = Z_0 + \sum_{i=1}^{n} \left(a^i Z_i + a^{\bar{i}} Z_{\bar{i}}\right), \tag{20}$$

and $\bar{a}^i = a^{\bar{i}}$. In virtue of $\varphi\xi = 0$ the last formula implies

$$\varphi Z_0 = -\sqrt{-1}\ \sum_{i=1}^{n} \left(a^i Z_i - a^{\bar{i}} Z_{\bar{i}}\right). \tag{21}$$

We point out the following relations between b_i, a^i and r

$$r = 1 + 2\sum_{i,j=1}^{n} a^i a^{\bar{j}} g_{i\,\bar{j}}, \qquad b_i = -\sum_{j=1}^{n} a^{\bar{j}} g_{i\,\bar{j}}, \tag{22}$$

they are consequences of $\eta(Z_A) = g(\xi, Z_A)$ and $g(\xi, \xi) = 1$.

Theorem 1. *Let $(\mathcal{M}, \varphi, \xi, \eta, g)$, $\dim \mathcal{M} = 2n + 1 \geqslant 3$ be a real-analytic almost cosymplectic manifold with Kälerian leaves. Then*

(a) *there is an open covering* $(\mathcal{U}_\iota)_{\iota\in I}$*, such that for each* $\mathcal{U}_\iota$ *there is a diffeomorphism*

$$f_\iota : U_\iota \to (-a,a)\times D',$$

where $(-a,a)$ *is an open interval,* $a>0$ *and* D' *is a domain in* $\mathbb{C}^n$*;*

(b) *on the set* U_ι*, the structure* (φ,ξ,η,g) *is described by the following local expressions*

$$\begin{aligned}
&\varphi\frac{\partial}{\partial t} = -\sqrt{-1}\,\sum_{i=1}^{n}\left(a^i\frac{\partial}{\partial z^i} - a^{\bar{i}}\frac{\partial}{\partial z^{\bar{i}}}\right),\\
&\varphi\frac{\partial}{\partial z^i} = \sqrt{-1}\,\frac{\partial}{\partial z^i}, \qquad \varphi\frac{\partial}{\partial z^{\bar{i}}} = -\sqrt{-1}\,\frac{\partial}{\partial z^{\bar{i}}}, \quad i=1,\dots,n,\\
&\eta = dt,\\
&\xi = \frac{\partial}{\partial t} + \sum_{i=1}^{n}\left(a^i\frac{\partial}{\partial z^i} + a^{\bar{i}}\frac{\partial}{\partial z^{\bar{i}}}\right),\\
&g = r\,dt^2 + 2\sum_{i=1}^{n}\left(b_i\,dt\,dz^i + b_{\bar{i}}\,dt\,dz^{\bar{i}}\right) + 2\sum_{i,j=1}^{n} g_{i\,\bar{j}}\,dz^i\,dz^{\bar{j}},
\end{aligned}$$

(c) *the coefficients* a^i, $a^{\bar{i}}$, b_i, $b_{\bar{i}}$, r, $g_{i\,\bar{j}}$ *are related by*

$$r = 1 + 2\sum_{i,j=1}^{n} a^i a^{\bar{j}} g_{i\,\bar{j}}, \qquad b_i = -\sum_{j=1}^{n} a^{\bar{j}} g_{i\,\bar{j}}, \quad a^{\bar{i}} = \bar{a}^i, \quad b_{\bar{i}} = \bar{b}_i,$$

and $(g_{i\,\bar{j}})$ *is* $n\times n$ *Hermitian matrix.*

3.3. **Hermitian structure of an almost cosymplectic CR manifold.** We start from an extension of the Levi-Civita connection to the complex connection in the complexified bundle $T_{\mathbb{C}}\mathcal{M}$. For a local section Z of $T_{\mathbb{C}}\mathcal{M}$

$$Z = \sum_{i=1}^{2n+1} f^i X_i,$$

where $(X_1,\dots,X_{2n+1})$ is a local frame of (real) vector fields on $\mathcal{M}$ and f^i are complex valued functions we define

$$\begin{aligned}
Z \mapsto \nabla Z = \nabla(\textstyle\sum_{i=1}^{2n+1} f^i X_i) = \\
= \left(\textstyle\sum_{i=1}^{2n+1} df^i \otimes X_i\right) + \left(\textstyle\sum_{i=1}^{2n+1} f^i \nabla X_i\right),
\end{aligned}$$

where $df = d(\Re f + \sqrt{-1}\Im f) = d\,(\Re f) + \sqrt{-1}d\,(\Im f)$. We have to of course verify that ∇Z is independent of the choice of a local real frame. This definition follows that

$$\begin{aligned}
&\nabla gZ = dg\otimes Z + g\nabla Z, \qquad \text{for a complex-valued function } g\\
&\nabla Z = \nabla(X+\sqrt{-1}Y) = \nabla X + \sqrt{-1}\nabla Y,
\end{aligned}$$

$X=\Re Z$, $Y=\Im Z$ are real and imaginary parts of Z. Now if Z is a section of the CR structure $\mathcal{D}'$ then there is a real vector field Y such that

$$Z = Y - \sqrt{-1}\varphi Y, \quad \eta(Y) = 0,$$

therefore

$$\begin{aligned}
\nabla_X Z &= \nabla_X Y - \sqrt{-1}\nabla_X\varphi Y = \\
&= ((\nabla_X Y - \eta(\nabla_X Y)\xi)) - \sqrt{-1}\varphi\nabla_X Y + \eta(\nabla_X Y)\xi + \sqrt{-1}g(A\varphi X, Y)\xi.
\end{aligned}$$

Let denote $Y' = \nabla_X Y - \eta(\nabla_X Y)\xi$, Y' is a tangential part of the derivative $\nabla_X Y$, i.e. $\eta(Y') = 0$, thus

$$(\nabla_X Y - \eta(\nabla_X Y)\xi) - \sqrt{-1}\varphi\nabla_X Y = Y' - \sqrt{-1}\varphi Y',$$

is a $\mathcal{D}'$-component of $\nabla_X Z$. Let

$$Z'' = \text{a } \mathcal{D}'\text{-component of } \nabla_X Z,$$

then

$$\nabla_X Z = Z'' + \eta(\nabla_X Y)\xi + \sqrt{-1}g(A\varphi X, Y)\xi.$$

Note that $\eta(\nabla_X Y)\xi = g(AX, Y)\xi$ according to the definition of A and $\eta(Y) = 0$:

$$\eta(\nabla_X Y)\xi + \sqrt{-1}g(A\varphi X, Y)\xi = \left(g(AX, Y) + \sqrt{-1}g(A\varphi X, Y)\right)\xi = g(X, A\bar{Z})\xi,$$

$\bar{Z}$ is the complex conjugate of Z. Finally

$$\nabla_X Z = Z'' + g(X, A\bar{Z})\xi.$$

Proposition 4. *The map*

$$Z \mapsto \nabla'_X Z = \nabla_X Z - g(X, A\bar{Z})\xi,$$

defines a complex connection in the CR structure $\mathcal{D}'$ - as a connection in a complex vector bundle. Moreover this connection is Hermitian with respect to a Hermitian metric H on $\mathcal{D}'$ defined by

$$H(Z, W) = g(Z, \bar{W}), \quad Z, W \in \Gamma(\mathcal{D}').$$

Proof. To prove that ∇' is Hermitian with respect to H we have to show that

$$XH(Z_1, Z_2) = H(\nabla'_X Z_1, Z_2) + H(Z_1, \nabla'_X Z_2),$$

for arbitrary X real vector field and arbitrary sections Z_1, Z_2 of $\mathcal{D}'$. From the definition:

$$XH(Z_1, Z_2) = Xg(Z_1, \bar{Z}_2) = g(\nabla_X Z_1, \bar{Z}_2) + g(Z_1, \nabla_X \bar{Z}_2),$$

note that

$$\begin{aligned}&\nabla_X \bar{Z}_2 = \overline{\nabla_X Z_2} = \overline{\nabla'_X Z_2} + \overline{g(X, A\bar{Z})\xi} = \overline{\nabla'_X Z_2} + g(X, AZ)\xi,\\ &g(\xi, \bar{Z}_2) = g(Z_1, \xi) = 0\end{aligned}$$

hence

$$g(\nabla_X Z_1, \bar{Z}_2) + g(Z_1, \nabla_X \bar{Z}_2) = g(\nabla'_X Z_1, \bar{Z}_2) + g(Z_1, \overline{\nabla'_X Z_2}) = H(\nabla'_X Z_1, Z_2) + H(Z_1, \nabla'_X Z_2).$$

□

It is very interesting topic to study the geometry of the manifold from the point of view of that canonical Hermitian structure. However these problems deserve its own attention so we stop here - as a starting point for the further investigations.

4. Almost cosymplectic (κ, μ, ν)- spaces

An almost contact metric structure of a model space $(-1, \mu, 0)$, $\mu = const$ can be realized as a left-invariant structure on a Lie group $\mathcal{G}$. If $\mathcal{G}$ is simply connected (and connected as all manifolds considered here are assumed to be connected) then $\mathcal{G}$ is diffeomorphic to $\mathbb{R}^{2n+1}$ and there is a frame of left-invariant vector fields $(Z, X_1, \ldots, X_n, Y_1, \ldots, Y_n)$, the basis of the Lie algebra $\mathfrak{g} = \mathfrak{g}(\mu)$ of $\mathcal{G}$, such that

$$\begin{aligned}&\xi = Z, \quad \eta(Z) = 1, \quad \eta(X_i) = \eta(Y_j) = 0,\ i, j = 1, \ldots, n,\\ &\varphi X_i = Y_i, \quad \varphi Y_i = -X_i,\ i = 1, \ldots, n,\end{aligned} \tag{23}$$

and the frame is orthonormal. Let $(t, x^1, \ldots, x^n, y^1, \ldots, y^n)$ be a global chart on $\mathbb{R}^{2n+1}$ then the vector fields X_i's, Y_j's are described explicitly as follows (in all cases $\xi = \partial/\partial t$ and $i = 1, \ldots, n$) [7]:

(1) $|\mu| < 2$, $\omega = \sqrt{1-\mu^2/4}$:

$$\begin{aligned} X_i &= (\cosh(\omega t) + \frac{\sinh(\omega t)}{\omega})\frac{\partial}{\partial x^i} - \frac{\mu\sinh(\omega t)}{2\omega}\frac{\partial}{\partial y^i}, \\ Y_i &= \frac{\mu\sinh(\omega t)}{2\omega}\frac{\partial}{\partial x^i} + (\cosh(\omega t) - \frac{\sinh(\omega t)}{\omega})\frac{\partial}{\partial y^i}, \end{aligned}$$

(2) $|\mu| = 2$ ($\omega = 0$):

$$X_i = (1+t)\frac{\partial}{\partial x^i} - \varepsilon\, t\frac{\partial}{\partial y^i}, \quad Y_i = \varepsilon\, t\frac{\partial}{\partial x^i} + (1-t)\frac{\partial}{\partial y^i}, \quad \varepsilon = \mu/2 = \pm 1,$$

(3) $|\mu| > 2$, $\omega = \sqrt{-1+\mu^2/4}$:

$$\begin{aligned} X_i &= (\cos(\omega t) + \frac{\sin(\omega t)}{\omega})\frac{\partial}{\partial x^i} - \frac{\mu\sin(\omega t)}{2\omega}\frac{\partial}{\partial y^i}, \\ Y_i &= \frac{\mu\sin(\omega t)}{2\omega}\frac{\partial}{\partial x^i} + (\cos(\omega t) - \frac{\sin(\omega t)}{\omega})\frac{\partial}{\partial y^i}, \end{aligned}$$

As we see the local descriptions are quite different depending on the value of μ. Note that algebras $\mathfrak{g}(\pm 2)$ are 'limits' (two-sided)

$$\mathfrak{g}(\pm 2) = \lim_{\mu\to\pm 2}\mathfrak{g}(\mu), \quad \mu \neq \pm 2.$$

Nevertheless the commutators of the vector fields can be described in a unique manner in all cases:

$$\mathfrak{g}(\mu): \quad [\xi, X_i] = X_i - \frac{\mu}{2}Y_i, \quad [\xi, Y_i] = \frac{\mu}{2}X_i - Y_i, \tag{24}$$

the other commutators vanish identically.

On the base of the relations (24) here we will provide a different local representation. Let again $\mathcal{M} = \mathbb{R}^{2n+1}$, $\mathcal{M} \ni p = (t, x^1, \ldots, x_n, y^1, \ldots, y_n)$ and we set

$$\begin{aligned} \xi &= \frac{\partial}{\partial t} + \sum_{i=1}^{n}(-x^i - \frac{\mu}{2}y^i)\frac{\partial}{\partial x^i} + \sum_{i=1}^{n}(\frac{\mu}{2}x^i + y^i)\frac{\partial}{\partial y^i}, \\ X_i &= \frac{\partial}{\partial x^i}, \qquad Y_i = \frac{\partial}{\partial y^i}, \quad i = 1, \ldots, n \end{aligned} \tag{25}$$

Direct computations show that such defined vector fields satisfy (24). Now, let identify $\mathbb{R}^{2n+1} \cong \mathbb{R}\times\mathbb{C}^n$ in the way that

$$\begin{aligned} &p = (t, x^1, \ldots, x^n, y^1, \ldots, y^n) = (t, z^1, \ldots, z^n) \in \mathbb{R}\times\mathbb{C}^n, \\ &z^1 = x^1 + \sqrt{-1}\, y^1, \ldots,\ z^n = x^n + \sqrt{-1}\, y^n, \end{aligned} \tag{26}$$

the vector fields

$$\begin{aligned} Z_0 &= \frac{\partial}{\partial t}, \\ Z_i &= \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\,\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial z^i}, \quad i = 1, \ldots, n \\ Z_{\bar{i}} &= \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial \bar{z}^i}, \quad i = 1, \ldots, n \end{aligned} \tag{27}$$

form a frame (in our case global) of vector fields of the complexified tangent bundle $T_{\mathbb{C}}\mathcal{M}$. According to (25), (26), (27) we have

$$\begin{aligned}\xi &= \frac{\partial}{\partial t} + \sum_{i=1}^{n}(-x^i - \frac{\mu}{2}y^i)\frac{\partial}{\partial x^i} + \sum_{i=1}^{n}(\frac{\mu}{2}x^i + y^i)\frac{\partial}{\partial y^i} = \\ &= \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}(-(z^i+\bar{z}^i) - \frac{\mu}{2}\frac{z^i-\bar{z}^i}{\sqrt{-1}})(\frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i}) + \\ &+ \frac{1}{2}\sum_{i=1}^{n}(\frac{\sqrt{-1}\mu}{2}(z^i+\bar{z}^i) + (z^i-\bar{z}^i))(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \bar{z}^i}).\end{aligned}$$

Providing similar computations for the (complexified) tensor fields φ, η, and g we obtain such expressions:

$$\begin{aligned}&\eta = dt, \\ &\xi = \frac{\partial}{\partial t} + \sum_{i=1}^{n}(-\bar{z}^i + \frac{\sqrt{-1}\mu}{2}z^i)\frac{\partial}{\partial z^i} + \sum_{i=1}^{n}(-z^i - \frac{\sqrt{-1}\mu}{2}\bar{z}^i)\frac{\partial}{\partial \bar{z}^i}, \\ &\sqrt{-1}\,\varphi\,\frac{\partial}{\partial t} = \sum_{i=1}^{n}(-\bar{z}^i + \frac{\sqrt{-1}\mu}{2}z^i)\frac{\partial}{\partial z^i} + \sum_{i=1}^{n}(z^i + \frac{\sqrt{-1}\mu}{2}\bar{z}^i)\frac{\partial}{\partial \bar{z}^i}, \\ &\varphi\frac{\partial}{\partial z^i} = \sqrt{-1}\frac{\partial}{\partial z^i}, \qquad \varphi\frac{\partial}{\partial \bar{z}^i} = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^i}, \qquad i = 1, \dots, n, \\ &ds^2 = r\,dt^2 + 2\sum_{i=1}^{n}(z^i + \frac{\sqrt{-1}\mu}{2}\bar{z}^i)dtdz^i + 2\sum_{i=1}^{n}(\bar{z}^i - \frac{\sqrt{-1}\mu}{2}z^i)dtd\bar{z}^i + 2\sum_{i=1}^{n}dz^i d\bar{z}^i, \\ &r = 1 + 2\sum_{i=1}^{n}|z^i + \frac{\sqrt{-1}\mu}{2}\bar{z}^i|^2.\end{aligned}$$

Recently D. Perrone has classified Riemannian homogeneous simply connected almost cosymplectic three-folds under an assumption that there is a group of isometries acting transitively and leaving the form η invariant [17].

Theorem 2. [17] *Let $(\mathcal{M}, \varphi, \xi, \eta, g)$ be a be a simply connected homogeneous* [1] *almost cosymplectic three-manifold. Then either $\mathcal{M}$ is a Lie group G equipped with a left invariant almost cosymplectic structure, or a Riemannian product of type $\mathbb{R}\times\mathcal{N}$, where $\mathcal{N}$ is a simply connected Kähler surface of constant curvature.*

From the list provided in [17] we are interested only in the case of non-cosymplectic three-manifolds and unimodular Lie groups as all Lie groups corresponding to the algebras $\mathfrak{g}(\mu)$ are unimodular:

- the universal covering $\tilde{E}(2)$ of the group of rigid motions of Euclidean 2-space, when $p > 0$,
- the group $E(1,1)$ of rigid motions of Minkowski 2-space when $p < 0$,
- the Heisenberg group H^3 when $p = 0$,

here p is a metric invariant of the classification defined by

$$p = \|\mathcal{L}_\xi h\| - 2\|h\|^2.$$

Corollary 1. *Let $(\mathcal{M}^3, \varphi, \xi, \eta, g)$ be a simply connected 3-dimensional almost cosymplectic $(-1, \mu, 0)$-space. Then $\mathcal{M}^3$ is a Lie group equipped with appropriate left invariant almost cosymplectic structure and*

- *$\mathcal{M}^3$ is a universal covering of the group of rigid motions of Euclidean 2-plane for $|\mu| > 2$,*

[1] It is assumed that η is invariant

- $\mathcal{M}^3$ *is a group of rigid motions of Minkowski 2-plane for* $|\mu| < 2$,
- $\mathcal{M}^3$ *is a Heisenberg group for* $|\mu| = 2$

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